

Shadow of Kerr-Taub-NUT black hole

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Abstract

The shadow of a rotating black hole with nonvanishing gravitomagnetic charge has been studied. It was shown that in addition to the angular momentum of black hole the gravitomagnetic charge term deforms the shape of the black hole shadow. From the numerical results we have obtained that for a given value of the rotation parameter, the presence of a gravitomagnetic charge enlarges the shadow and reduces its deformation with respect to the spacetime without gravitomagnetic charge. Finally we have studied the capture cross section for massive particles by black hole with the nonvanishing gravitomagnetic charge.

Keywords Photon motion Shadow of Black hole NUT spacetime

1 Introduction

At present there is no any observational evidence for the existence of gravitomagnetic monopole, i.e. so-called NUT (Newman et al. 1963) parameter. Therefore study the motion of massless and the massive test particles and particle acceleration mechanisms in

NUT spacetime may provide new tool for studying new important general relativistic effects which are associated with nondiagonal components of the metric tensor and have no Newtonian analogues (See, e.g. Nour-Zonoz (2004); Kagramanova et al. (2008); Morozova and Ahmedov (2009) where solutions for electromagnetic waves and interferometry in spacetime with NUT parameter have been studied.). Kerr-Taub-NUT spacetime with Maxwell and dilation fields is investigated by Aliev et al. (2008). In our preceding papers (Morozova et al. 2008; Abdujabbarov et al. 2008) we have studied the plasma magnetosphere around a rotating, magnetized neutron star and charged particle motion around compact objects immersed in external magnetic field in the presence of the NUT parameter. The Penrose process in the spacetime of rotating black hole with nonvanishing gravitomagnetic charge has been considered by Abdujabbarov et al. (2011).

Analytic treatment of geodesics of electrically and magnetically charged test particles in the Reissner-Nordström and Taub-NUT space-times have been considered by Grunau and Kagramanova (2011); Kagramanova et al. (2010); Hackmann and Lämmerzahl (2012). Kubizňák and coauthors have considered the parallel-propagated frame along null geodesics in higher-dimensional black hole spacetimes (Kubizňák et al. 2009). The parallel transport equations in the higher-dimensional Kerr-NUT-(A)dS spacetimes have been solved by Connel et al. (2008); Frolov and Krtouš (2011); Krtouš et al. (2008). Magnetized black hole on the Taub-NUT instanton has been considered by the authors of the paper (Nedkova and Yazadjiev 2012). Some geometrical properties of Taub-NUT metric, such as asymptotically flatness and variational principle have been analyzed by Virmani (2011).

In this paper we study photon orbits around the rotating black hole with nonvanishing gravitomagnetic charge as well as black hole shadow in the vicinity

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of Kerr-Taub-NUT spacetime. The paper organized as follows: in Sec. 2, we review the basic aspects of the geometry and the geodesics of the Kerr-Taub-NUT black hole. In Sec. 3, we obtain the shadows of black holes for the different values of the angular momentum and gravitomagnetic charge of black hole. In Sect. 4 we study the capture cross section for massive particles by black hole with nonvanishing gravitomagnetic charge. Finally, in Sec. 5, we discuss the results found. Throughout the paper, we use a space-like signature $(-, +, +, +)$ and a system of units in which $G = 1 = c$ (However, for those expressions with an astrophysical application we have written the speed of light explicitly.). Greek indices are taken to run from 0 to 3 and Latin indices from 1 to 3; covariant derivatives are denoted with a semi-colon and partial derivatives with a comma.

2 Photon motion around Kerr-Taub NUT black holes

Here we will study massless particles motion in the vicinity of a black hole of mass M in the presence of gravitomagnetic charge described by the space-time metric (Newman et al. 1963; Morozova et al. 2008; Abdujabbarov et al. 2008):

$$ds^2 = -\frac{1}{\Sigma} (\Delta - a^2 \sin^2 \theta) dt^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{1}{\Sigma} [(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta] d\phi^2 + \frac{2}{\Sigma} (\Delta\chi - a(\Sigma + a\chi) \sin^2 \theta) dt d\phi, \quad (1)$$

where notations Δ, Σ , and χ are defined as

$$\Delta = r^2 + a^2 - 2Mr - l^2,$$

$$\Sigma = r^2 + (l + a \cos \theta)^2,$$

$$\chi = a \sin^2 \theta - 2l \cos \theta,$$

here a is the specific angular momentum per total mass of black hole $M(a = J/M)$ and l is the gravitomagnetic charge. The event horizon is determined by largest root of the equation $\Delta = 0$, given by

$$r_+ = M + \sqrt{M^2 - a^2 + l^2}. \quad (2)$$

Consider a black hole placed between a source of light and an observer. Then the light reaches the observer after being deflected by the black hole gravitational field. But some part of the deflected light with

small impact parameters emitted by the source falling into the black hole. This phenomena result a dark figure in the map of the space called the shadow. The boundary of this shadow defines the shape of a black hole (see e.g. Amarilla and Eiroa (2012)). The study of the geodesic structure around black hole is very important to obtain the apparent shape. The Hamilton-Jacobi equation determines the geodesics for a given space-time geometry:

$$\frac{\partial S}{\partial \lambda} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, \quad (3)$$

where λ is an affine parameter along the geodesics, $g_{\mu\nu}$ are the components of the metric tensor and S is the Jacobi action. If the problem is separable (the separable problem of Hamilton-Jacobi equation in Kerr-Taub-NUT spacetime has been studied by Dadhich and Turakulov (2002)), the Jacobi action S can be written in the form

$$S = \frac{1}{2} m^2 \lambda - \mathcal{E} t + \mathcal{L} \phi + S_r(r) + S_\theta(\theta), \quad (4)$$

where m is the mass of a test particle. The second term in the right hand side is related to the conservation of energy \mathcal{E} , while the third term is related to the conservation of the angular momentum \mathcal{L} in the direction of the axis of symmetry. In the case of null geodesics, we have that $m = 0$, and from the Hamilton-Jacobi equation, the following equations of motion are obtained:

$$\Sigma \frac{dt}{d\lambda} = \frac{r^2 + a^2 + l^2}{\Delta} [(r^2 + a^2 + l^2) \mathcal{E} - a \mathcal{L}] + \frac{\chi}{\sin^2 \theta} (\mathcal{L} - \chi \mathcal{E}), \quad (5)$$

$$\Sigma \frac{d\phi}{d\lambda} = \frac{a}{\Delta} [(r^2 + a^2 + l^2) \mathcal{E} - a \mathcal{L}] + \frac{1}{\sin^2 \theta} (\mathcal{L} - \chi \mathcal{E}), \quad (6)$$

$$\Sigma \frac{dr}{d\lambda} = \sqrt{\mathcal{R}}, \quad (7)$$

$$\Sigma \frac{d\theta}{d\lambda} = \sqrt{\Theta}, \quad (8)$$

where the functions $\mathcal{R}(r)$ and $\Theta(\theta)$ are defined by

$$\begin{aligned} \mathcal{R} &= [(r^2 + a^2 + l^2) \mathcal{E} - a \mathcal{L}]^2 - \Delta [\mathcal{K} + (\mathcal{L} - a \mathcal{E})^2] (9) \\ \Theta &= \mathcal{K} + \cos^2 \theta \left[\left(a^2 - \frac{4l^2}{\sin^2 \theta} \right) \mathcal{E}^2 - \frac{\mathcal{L}^2}{\sin^2 \theta} \right] \\ &\quad + 4l \mathcal{E} \cos \theta \left(\mathcal{E} a - \frac{\mathcal{L}}{\sin^2 \theta} \right), \end{aligned} \quad (10)$$

and \mathcal{K} is a constant of separation. Defining the effective potential for massless particles as $(dr/d\lambda)^2 = V_{\text{eff}}$ one may study the radial motion of photons in the presence

of gravitomagnetic charge. In the Fig. 1 the radial dependence of the effective potential of radial photon motion is shown. From the figure it is seen that with the increase of the gravitomagnetic charge the shape of the effective potential is going to shift to the observer at infinity. This corresponds to increasing the event horizon of the Kerr-Taub-NUT black hole. Moreover, one may conclude from the Fig. 1 that with the increase of the gravitomagnetic charge the circular photon orbits become unstable.

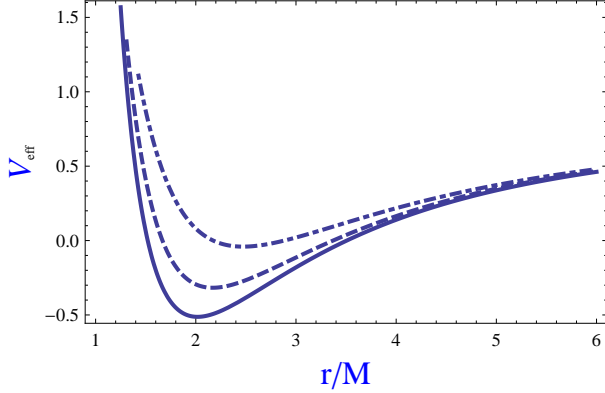


Fig. 1 The radial dependence of the effective potential of radial motion of the massless particles for the different values of the gravitomagnetic charge: solid line for $l/M = 0.1$, dashed line for $l/M = 0.5$, and dot-dashed line for $l = 0.9$.

The propagation of light in the Kerr-Taub-NUT spacetime is totally determined by the equations (5)–(8). Two impact parameters can be expressed in terms of the constants of motion \mathcal{E} , \mathcal{L} and the Carter constant \mathcal{K} and characterize the light rays. One can easily define impact parameters for general orbits around the black hole as $\xi = \mathcal{L}/\mathcal{E}$ and $\eta = \mathcal{K}/\mathcal{E}^2$. The equation (7) can be used to derive the orbits with constant r in order to obtain the boundary of the shadow of the black hole. These orbits satisfy the conditions fulfilled by the values of the impact parameters that determine the contour of the shadow $\mathcal{R}(r) = 0 = d\mathcal{R}(r)/dr$,

$$\xi(r) = \frac{a^2(1+r) + r^2(r-3) + l^2(1-3r)}{a(1-r)}, \quad (11)$$

$$\eta(r) = a^{-2}(r-1)^{-2} \left\{ r^3[4a^2 - r(r-3)^2] - l^2[4r^2a^2 + (1-3r)(l^2(1-3r) - 6r^2 + 4a^2r + 2r^3)] \right\} \quad (12)$$

Note that the corresponding values being valid in the Kerr geometry can be obtained in the limiting case when $l = 0$.

3 Kerr-Taub-NUT black hole shadow

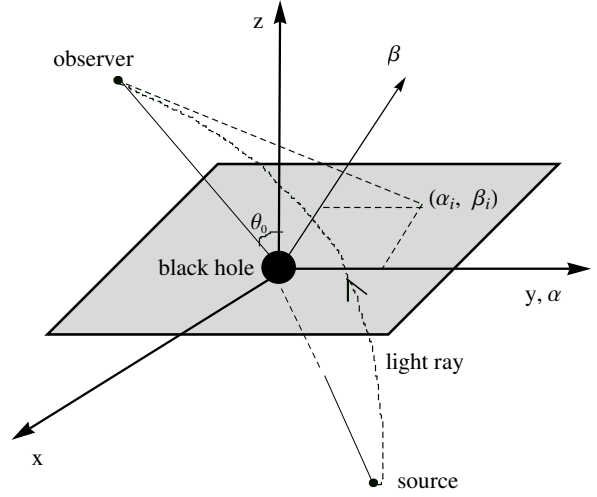


Fig. 2 The schematic geometry of the gravitational lens. An observer far away from the black hole, can set up a reference coordinate system (x, y, z) with the black hole at the origin. The Boyer-Lidquist coordinates coincide with this system only at infinity. The tangent vector to an incoming light ray defines a straight line, which intersects the $\alpha - \beta$ plane at the point (α_i, β_i) .

Using the celestial coordinates one can easily describe the shadow (see for example Vázquez and Esteban (2004)):

$$\alpha = \lim_{r_0 \rightarrow \infty} \left(-r_0^2 \sin \theta_0 \frac{d\phi}{dr} \right), \quad (13)$$

and

$$\beta = \lim_{r_0 \rightarrow \infty} r_0^2 \frac{d\theta}{dr}, \quad (14)$$

since here an observer far away from the black hole is considered $r_0 \rightarrow \infty$, θ_0 is the angular coordinate of the observer, i.e. the inclination angle between the rotation axis of the black hole and the line of sight of the observer. The geometry of the new introduced coordinates is shown in Fig. 2. The coordinates α and β are the apparent perpendicular distances of the image as seen from the axis of symmetry and from its projection on the equatorial plane, respectively.

Calculating $d\phi/dr$ and $d\theta/dr$ using the spacetime metric (1) and taking the limit of a far away observer one can obtain celestial coordinates as

$$\alpha = -\xi \csc \theta_0, \quad (15)$$

and

$$\beta = \pm \left[\eta + \left(a^2 - \frac{4l^2}{\sin^2 \theta_0} \right) \cos^2 \theta_0 - \xi^2 \cot^2 \theta_0 + 4l \cos \theta_0 \left(a - \frac{\xi}{\sin^2 \theta_0} \right) \right]^{1/2}, \quad (16)$$

where expressions (6), (7), and (8) were used to calculate $d\theta/dr$ and $d\phi/dr$. These equations have implicitly the same form as for the Kerr metric, with the new radial functions ξ and η given by Eqs. (11) and (12) (a detailed calculation of the values of ξ and η , and the expressions of the celestial coordinates α and β as a function of the constants of motion for the Kerr geometry, are given in (Vázquez and Esteban 2004)).

In the case of rotating black hole, one may introduce two observables which approximately characterize the apparent shape. First one should approximate the apparent shape by a circle passing through three points which are located at the top position, bottom, and the most right end of the shadow as shown in Fig. 3. the radius R_s of the shadow is defined by the radius of this circle. One can also define the distortion parameter δ_s of the black hole shadow by $\delta_s = D_{cs}/R_s$. Two variables (R_s and δ_s) can be interpreted as observable in astronomical observation (Hioki and Maeda 2009).

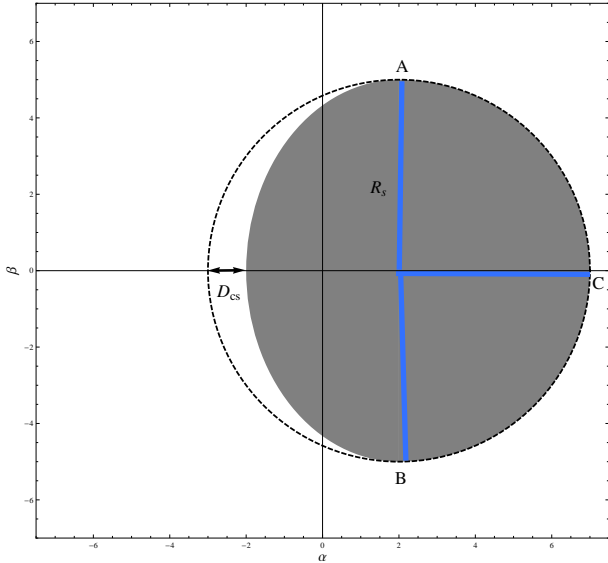


Fig. 3 The observables for the apparent shape of a rotating black hole are the radius R_s and the distortion parameter $\delta_s = D_{cs}/R_s$. Here D_{cs} is the difference between the left endpoints of the circle and of the shadow.

If the observer is situated in the equatorial plane of the black hole, then the inclination angle is $\theta_0 = \pi/2$. The gravitational effects on the shadow, which grow with θ_0 , are larger. In this case, one has

$$\alpha = -\xi, \quad (17)$$

and

$$\beta = \pm \sqrt{\eta}. \quad (18)$$

For the visualization of the shape of the black hole shadow one needs to plot β vs α . In Fig. 4, we show the contour of the shadows of black holes. From the plots one can see that the presence of the gravitomagnetic charge will increase the effective size of the shadow. We plot the shapes of the shadow of black hole with the gravitomagnetic charge for the different values of black hole angular momentum a : $a/M = 0.5$, $a/M = 0.7$, $a/M = 0.8$, and $a/M = 0.99$. One can compare the effect of the NUT parameter and the black hole angular momentum on modification of the shape of the shadow of black hole. It appears they have opposite effects on black hole shadow size. The gravitomagnetic charge increases the size of the shadow shape while black hole's angular momentum decreases its size.

The observable R_s can be calculated from the equation

$$R_s = \frac{(\alpha_t - \alpha_r)^2 + \beta_t^2}{2|\alpha_t - \alpha_r|},$$

and the observable δ_s is given by

$$\delta_s = \frac{\tilde{\alpha}_p - \alpha_p}{R_s},$$

where $(\tilde{\alpha}_p, 0)$ and $(\alpha_p, 0)$ are the points where the reference circle and the contour of the shadow cut the horizontal axis at the opposite side of $(\alpha_r, 0)$, respectively. In Fig. 5, the observables R_s and δ_s are shown as functions of the gravitomagnetic charge l . From the dependence of R_s from the NUT parameter one can again see that the gravitomagnetic charge forces to increase the size of the black hole shadow. The dependence of δ_s from NUT charge shows that gravitomagnetic charge forces to shadow to get the shape of circle than ellipse. In the case of rotation, with the increase of black hole's angular momentum the shape of black hole shadow takes form of ellipse rather than circle.

4 PARTICLE CAPTURE CROSS SECTIONS FOR BLACK HOLE WITH NUT CHARGE

In this section we will study the pure effect of NUT parameter assuming that the the angular momentum of the black hole is equal to zero. It has been shown in the paper of Abdujabbarov et al. (2008) that variables in the Hamilton-Jacobi equation for the particle motion around NUT black hole can be separated in the equatorial plane. In the space-time metric (1) we assume

that the central object is non-rotating and particles are confined at the equatorial plane ($a = 0$, and $\theta = \pi/2$).

It was first shown by Zimmerman and Shahir (1989) for the spherical symmetric case (NUT spacetime) and later in the paper of Bini et al. (2003) for the axial symmetric case (Kerr-Taub-NUT spacetime) that the orbits of the test particles are confined to a cone with the opening angle θ given by $\cos\theta = 2\mathcal{E}l/\mathcal{L}$. It also follows that in this case the equations of motion on the cone depend on l only via l^2 (Bini et al. 2003; Abdujabbarov et al. 2008).

The main point is that the small value for the upper limit for gravitomagnetic moment has been obtained by comparing theoretical results with experimental data as (i) $l/M \leq 10^{-24}$ from the gravitational microlensing (Rahvar and Habibi 2004), (ii) $l/M \leq 1.5 \cdot 10^{-18}$ from the interferometry experiments on ultra-cold atoms (Morozova and Ahmedov 2009), (iii) and similar limit has been obtained from the experiments on Mach-Zehnder interferometer (Kagramanova et al. 2008) (here M is the total mass of central gravitating object).

Due to the smallness of the gravitomagnetic charge let us consider the motion in the quasi-equatorial plane when the motion in θ direction changes as $\theta = \pi/2 + \delta\theta(t)$, where $\delta\theta(t)$ is the term of first order in l , then it is easy to expand the trigonometric functions as $\sin\theta = 1 - \delta\theta^2(t)/2 + \mathcal{O}(\delta\theta^4(t))$ and $\cos\theta = \delta\theta(t) - \mathcal{O}(\delta\theta^3(t))$. Neglecting the small terms $\mathcal{O}(\delta\theta^2(t))$, one can write $\Sigma = r^2 + l^2$, $\Delta = r^2 - 2Mr - l^2$, and $\chi = 0$ and consequently the equation of motion for the radial motion takes the following form

$$r^4 \left(\frac{dr}{d\lambda} \right)^2 = R(r) = [\mathcal{E}^2 - 1 - 2U_{\text{eff}}(r, l, \mathcal{L})] r^4, \quad (19)$$

where \mathcal{E} and \mathcal{L} is the energy and angular momentum of the particle per unit of its mass and the quantity

$$U_{\text{eff}}(r, l, \mathcal{L}) = -\frac{l^2 + Mr}{\Sigma} + \frac{\Delta \mathcal{L}^2}{2\Sigma^2} \quad (20)$$

can be interpreted as effective potential of the radial motion of the test particle at equatorial plane. The radial dependence of the effective potential of radial motion of the massive particles for the different values of the gravitomagnetic charge is presented in Fig. 6.

Assuming that the uncharged particle is moving slowly at infinity, i.e. $\mathcal{E} \simeq 1$ one can easily rewrite the expression (19) in the following form:

$$R(\rho) = \rho^3 + (\tilde{l} - \tilde{\mathcal{L}}) \rho^2 + \tilde{\mathcal{L}} \rho + \frac{\tilde{l}\tilde{\mathcal{L}}}{2}, \quad (21)$$

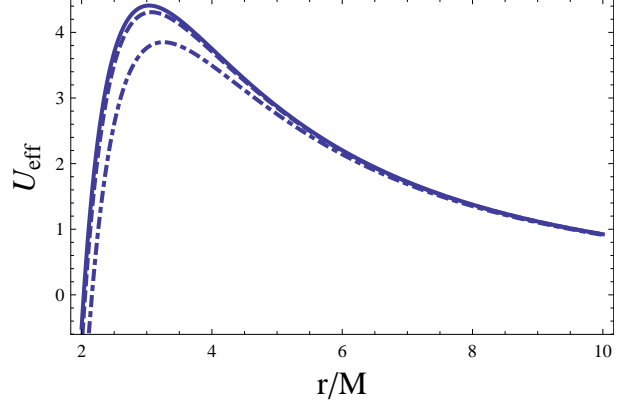


Fig. 6 The radial dependence of the effective potential of radial motion of the massive particles for the different values of the gravitomagnetic charge: solid line for $l/M = 0.1$, dashed line for $l/M = 0.5$, and dot-dashed line for $l/M = 0.9$.

where

$$\rho = \frac{r}{M}, \quad \tilde{l} = \left(\frac{l}{M} \right)^2, \quad \tilde{\mathcal{L}} = \left(\frac{\mathcal{L}}{M} \right)^2.$$

Gravitational capture of the particle occurs for $\mathcal{L} \leq \mathcal{L}_{\text{cr}}$. For the $\mathcal{L} = \mathcal{L}_{\text{cr}}$ orbit spirals into a circular orbit, the radius of which is determined by the value of the multiple root of the polynomial (21), i.e. discriminant of the later should vanish. Now it is easy to find the expression for \mathcal{L}_{cr} as

$$\mathcal{L}_{\text{cr}}^2 = 16M^2 - 6l^2 - \frac{13l^4}{16M^2}. \quad (22)$$

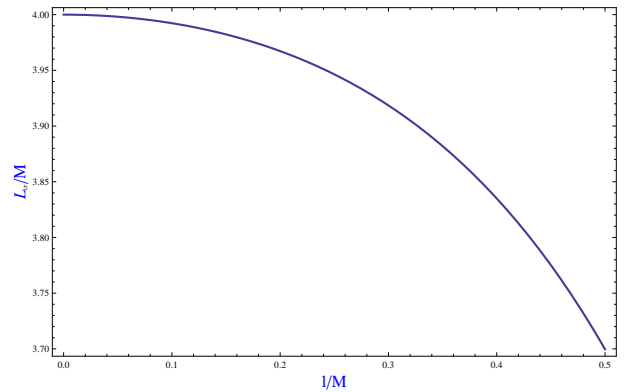


Fig. 7 The dependence of the critical angular momentum for capturing by central black hole from gravitomagnetic monopole momentum.

In the Fig.7 the dependence of \mathcal{L}_{cr} from dimensionless NUT parameter is presented. The dependence shows that the presence of the gravitomagnetic charge decreases the capture cross section for particles by black hole.

5 Conclusion

In this paper, we have studied the shadow of black hole with nonvanishing gravitomagnetic charge and analyzed how the shadow of the black hole will be distorted by the presence of the NUT parameter. From the numerical results we have obtained that the NUT parameter forces to increase the size of the black hole shadow. The dependence of the distortion parameter δ_s from the NUT charge shows that the gravitomagnetic charge forces black hole's shadow to get the shape of circle than ellipse. We have also studied the capture cross section for massive particles by black hole with nonvanishing gravitomagnetic charge and found its strong dependence from the NUT parameter.

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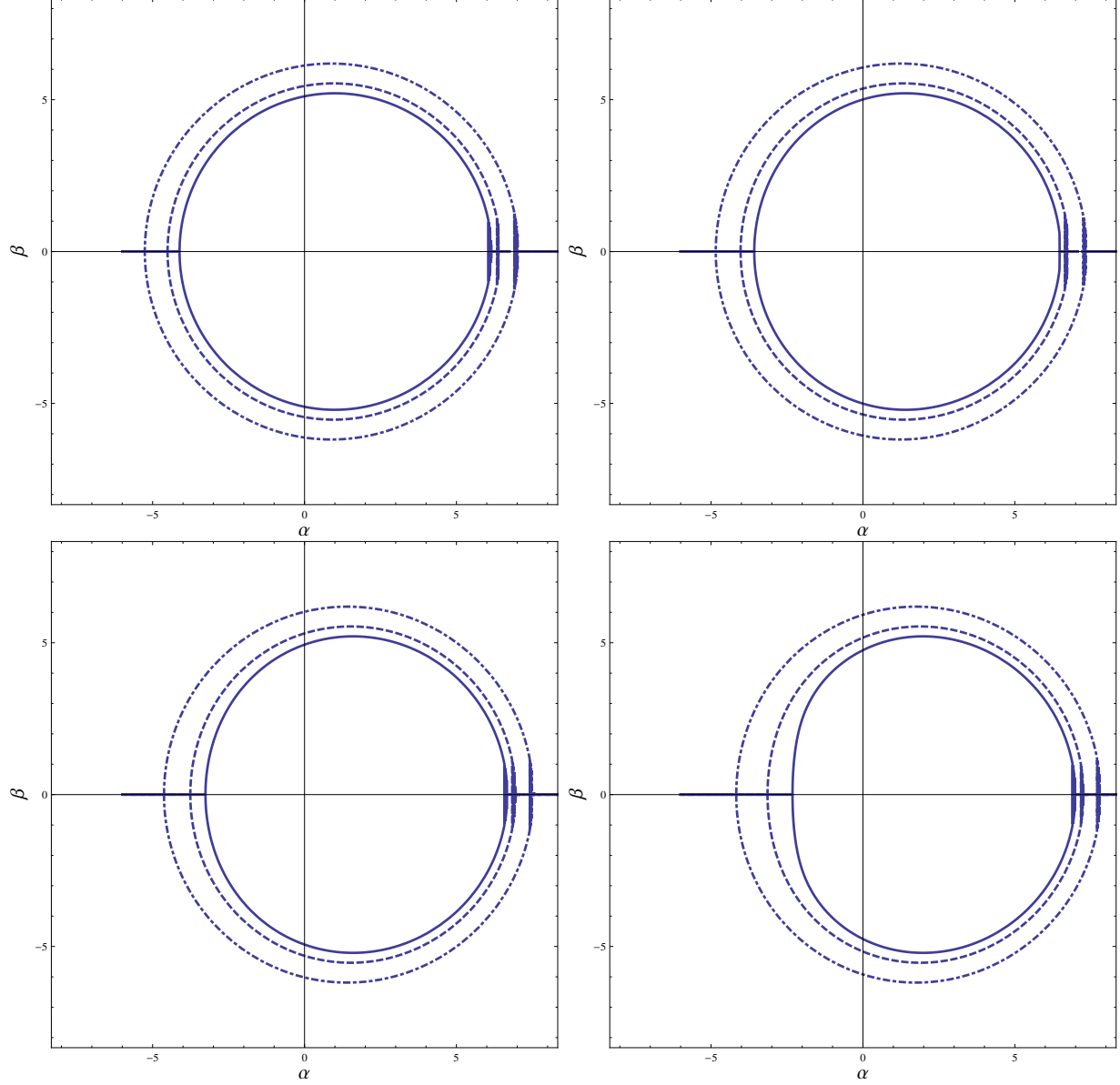


Fig. 4 Silhouette of the shadow cast by a Kerr-Taub-NUT black hole situated at the origin of coordinates with inclination angle $\theta = \pi/2$, having a black hole angular momentum a and a NUT charge l . Upper row, left: $a/M = 0.5$, $l/M = 0.1$ (solid line), $l/M = 0.5$ (dashed line), and $l/M = 0.9$ (dashed-dotted line). Upper row, right: $a/M = 0.7$, $l/M = 0.1$ (solid line), $l/M = 0.5$ (dashed line), and $l/M = 0.9$ (dashed-dotted line). Lower row, left: $a/M = 0.8$, $l/M = 0.1$ (solid line), $l/M = 0.5$ (dashed line), and $l/M = 0.9$ (dashed-dotted line). Lower row, right: $a/M = 0.99$, $l/M = 0.1$ (solid line), $l/M = 0.5$ (dashed line), and $l/M = 0.9$ (dashed-dotted line). The shadow corresponds to each curve and the region inside it.

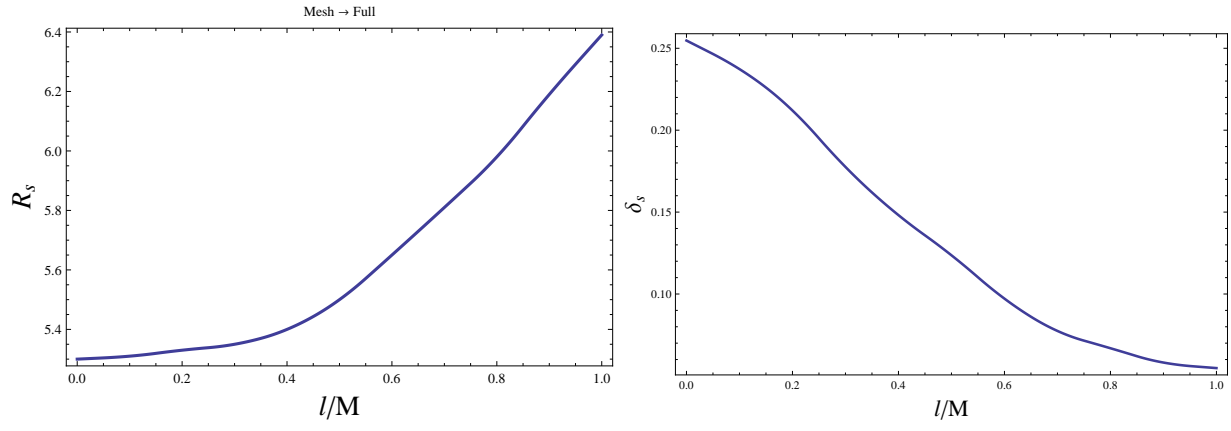


Fig. 5 Observables R_s and δ_s as functions of the NUT charge, corresponding to the shadow of a black hole situated at the origin of coordinates with inclination angle $\theta = \pi/2$ and spin parameters $a = 0.99$.

References

- Abdujabbarov, A.A., Ahmedov, B.J., Kagramanova, V.G.: Gen. Rel. Grav. **40**, 2515 (2008)
- Abdujabbarov, A.A., Ahmedov, B.J., Shaymatov, S.R., Rakhmatov, A.S.: Astrophys Space Sci **334**, 237 (2011)
- Aliev, A. N., Cebeci, H., Dereli, T.: Phys. Rev. D **77**, 124022 (2008)
- Amarilla, L., Eiroa, E. F.: Phys Rev. D **85**, 064019 (2012)
- Bini, D., Cherubini, C., Janzen, R., Mashhoon, B.: Class. Quantum Grav. **20**, 457 (2003)
- Connell, P., Frolov, V., Kubizňák, D.: Phys Rev. D **78**, 024042 (2008)
- Dadhich, N., Turakulov, Z.Ya.: Class. Quantum Grav. **19**, 2765 (2002)
- Frolov, V., Krtouš, P.: Phys Rev. D **83**, 024016 (2011).
- Grunau, S., Kagramanova, V.: Phys Rev. D **83**, 044009 (2011)
- Hackmann, E., Lämmerzahl, C.: Phys Rev. D **85**, 044049 (2012)
- Hioki K., Maeda, K.I.: Phys. Rev. D **80**, 024042 (2009)
- Kagramanova, V., Kunz, J., Lämmerzahl, C.: Class. Quantum Grav. **25**, 105023 (2008)
- Kagramanova, V., Kunz, J., Hackmann, E., Lämmerzahl, C.: Phys Rev. D **81** 124044 (2010)
- Krtouš, P., Frolov, V., Kubizňák, D.: Phys Rev. D **78**, 064022 (2008)
- Kubizňák, D., Frolov, V., Krtouš, P., Connell, P.: Phys Rev. D **79**, 024018 (2009)
- Morozova, V.S., Ahmedov, B.J.: Int. J. Mod. Phys. D **18**, 107 (2009)
- Morozova, V. S., Ahmedov, B.J., Kagramanova, V.G.: Astrophys. J. **684**, 1359 (2008)
- Nedkova, P., Yazadjiev, S.: Phys Rev. D **85**, 064021 (2012)
- Newman, E., Tamburino, L., Unti, T.: J. Math. Phys. **4**, 915 (1963)
- Nouri-Zonoz, M.: Class. Quantum Grav. **21**, 471 (2004)
- Rahvar, S., Habibi, F.: Astroph. J., **610**, 673 (2004)
- Virmani, A.: Phys Rev. D **84**, 064034 (2011)
- Vázquez S., Esteban, E.: Nuovo Cim. **119B**, 489 (2004)
- Zimmerman, R., Shahir, B.: Gen. Relativ. Gravit. **21**, 821 (1989)

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of Kerr-Taub-NUT spacetime. The paper organized as follows: in Sec. 2, we review the basic aspects of the geometry and the geodesics of the Kerr-Taub-NUT black hole. In Sec. 3, we obtain the shadows of black holes for the different values of the angular momentum and gravitomagnetic charge of black hole. In Sect. 4 we study the capture cross section for massive particles by black hole with nonvanishing gravitomagnetic charge. Finally, in Sec. 5, we discuss the results found. Throughout the paper, we use a space-like signature $(-, +, +, +)$ and a system of units in which $G = 1 = c$ (However, for those expressions with an astrophysical application we have written the speed of light explicitly.). Greek indices are taken to run from 0 to 3 and Latin indices from 1 to 3; covariant derivatives are denoted with a semi-colon and partial derivatives with a comma.

2 Photon motion around Kerr-Taub NUT black holes

Here we will study massless particles motion in the vicinity of a black hole of mass M in the presence of gravitomagnetic charge described by the space-time metric (Newman et al. 1963; Morozova et al. 2008; Abdujabbarov et al. 2008):

$$ds^2 = -\frac{1}{\Sigma} (\Delta - a^2 \sin^2 \theta) dt^2 + \Sigma \left(\frac{dr^2}{\Delta} + d\theta^2 \right) + \frac{1}{\Sigma} [(\Sigma + a\chi)^2 \sin^2 \theta - \chi^2 \Delta] d\phi^2 + \frac{2}{\Sigma} (\Delta\chi - a(\Sigma + a\chi) \sin^2 \theta) dt d\phi, \quad (1)$$

where notations Δ, Σ , and χ are defined as

$$\Delta = r^2 + a^2 - 2Mr - l^2,$$

$$\Sigma = r^2 + (l + a \cos \theta)^2,$$

$$\chi = a \sin^2 \theta - 2l \cos \theta,$$

here a is the specific angular momentum per total mass of black hole ($a = J/M$) and l is the gravitomagnetic charge. The event horizon is determined by largest root of the equation $\Delta = 0$, given by

$$r_+ = M + \sqrt{M^2 - a^2 + l^2}. \quad (2)$$

When a black hole is placed between a source of light and an observer, the light reaches the observer after being deflected by the black hole gravitational field; but some part of the photons emitted by the source, those

with small impact parameters, end up falling into the black hole, not reaching the observer, giving as result a dark zone in the sky called the shadow. The apparent shape of a black hole is thus defined by the boundary of the shadow (see e.g. Amarilla and Eiroa (2012)). In order to obtain the apparent shape, we need to study the geodesic structure around black hole. The Hamilton-Jacobi equation determines the geodesics for a given space-time geometry:

$$\frac{\partial S}{\partial \lambda} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, \quad (3)$$

where λ is an affine parameter along the geodesics, $g_{\mu\nu}$ are the components of the metric tensor and S is the Jacobi action. If the problem is separable (the separable problem of Hamilton-Jacobi equation in Kerr-Taub-NUT spacetime has been studied by Dadhich and Turakulov (2002)), the Jacobi action S can be written in the form

$$S = \frac{1}{2} m^2 \lambda - \mathcal{E} t + \mathcal{L} \phi + S_r(r) + S_\theta(\theta), \quad (4)$$

where m is the mass of a test particle. The second term in the right hand side is related to the conservation of energy \mathcal{E} , while the third term is related to the conservation of the angular momentum \mathcal{L} in the direction of the axis of symmetry. In the case of null geodesics, we have that $m = 0$, and from the Hamilton-Jacobi equation, the following equations of motion are obtained:

$$\Sigma \frac{dt}{d\lambda} = \frac{r^2 + a^2 + l^2}{\Delta} [(r^2 + a^2 + l^2) \mathcal{E} - a \mathcal{L}] + \frac{\chi}{\sin^2 \theta} (\mathcal{L} - \chi \mathcal{E}), \quad (5)$$

$$\Sigma \frac{d\phi}{d\lambda} = \frac{a}{\Delta} [(r^2 + a^2 + l^2) \mathcal{E} - a \mathcal{L}] + \frac{1}{\sin^2 \theta} (\mathcal{L} - \chi \mathcal{E}), \quad (6)$$

$$\Sigma \frac{dr}{d\lambda} = \sqrt{\mathcal{R}}, \quad (7)$$

$$\Sigma \frac{d\theta}{d\lambda} = \sqrt{\Theta}, \quad (8)$$

where the functions $\mathcal{R}(r)$ and $\Theta(\theta)$ are defined by

$$\begin{aligned} \mathcal{R} &= [(r^2 + a^2 + l^2) \mathcal{E} - a \mathcal{L}]^2 - \Delta [\mathcal{K} + (\mathcal{L} - a \mathcal{E})^2] (9) \\ \Theta &= \mathcal{K} + \cos^2 \theta \left[(a^2 - \frac{4l^2}{\sin^2 \theta}) \mathcal{E}^2 - \frac{\mathcal{L}^2}{\sin^2 \theta} \right] \\ &\quad + 4l \mathcal{E} \cos \theta \left(\mathcal{E} a - \frac{\mathcal{L}}{\sin^2 \theta} \right), \end{aligned} \quad (10)$$

and \mathcal{K} is a constant of separation. Defining the effective potential for massless particles as $(dr/d\lambda)^2 = V_{\text{eff}}$ one may study the radial motion of photons in the presence

of gravitomagnetic charge. In the Fig. 1 the radial dependence of the effective potential of radial photon motion is shown. From the figure it is seen that with the increase of the gravitomagnetic charge the shape of the effective potential is going to shift to the observer at infinity. This corresponds to increasing the event horizon of the Kerr-Taub-NUT black hole. Moreover, one may conclude from the Fig. 1 that with the increase of the gravitomagnetic charge the circular photon orbits become unstable.

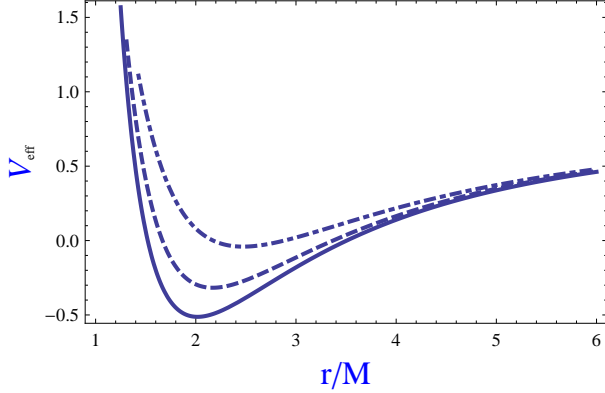


Fig. 1 The radial dependence of the effective potential of radial motion of the massless particles for the different values of the gravitomagnetic charge: solid line for $l/M = 0.1$, dashed line for $l/M = 0.5$, and dot-dashed line for $l = 0.9$.

The equations (5)–(8) determine the propagation of light in the Kerr-Taub-NUT spacetime. The light rays are, in general, characterized by two impact parameters, which can be expressed in terms of the constants of motion \mathcal{E} , \mathcal{L} and the Carter constant \mathcal{K} . Combining these quantities we define as usual $\xi = \mathcal{L}/\mathcal{E}$ and $\eta = \mathcal{K}/\mathcal{E}^2$, which are the impact parameters for general orbits around the black hole. We use Eq. (7) to derive the orbits with constant r in order to obtain the boundary of the shadow of the black hole. These orbits satisfy the conditions $\mathcal{R}(r) = 0 = d\mathcal{R}(r)/dr$, which are fulfilled by the values of the impact parameters that determine the contour of the shadow, namely,

$$\xi(r) = \frac{a^2(1+r) + r^2(r-3) + l^2(1-3r)}{a(1-r)}, \quad (11)$$

$$\eta(r) = a^{-2}(r-1)^{-2} \left\{ r^3[4a^2 - r(r-3)^2] - l^2[4r^2a^2 + (1-3r)(l^2(1-3r) - 6r^2 + 4a^2r + 2r^3)] \right\} \quad (12)$$

Note that the corresponding values being valid in the Kerr geometry can be obtained in the limiting case when $l = 0$.

3 Kerr-Taub-NUT black hole shadow

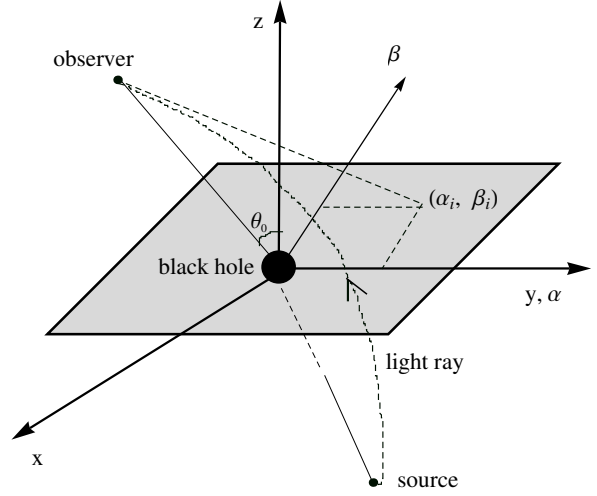


Fig. 2 The geometry of the gravitational lens. An observer far away from the black hole, can set up a reference coordinate system (x, y, z) with the black hole at the origin. The Boyer-Lidquist coordinates coincide with this system only at infinity. The reference frame is chosen so that, as seen from infinity, the black hole is rotating around the z axis. In this system, the line joining the origin with the observer is normal to the $\alpha - \beta$ plane. The tangent vector to an incoming light ray defines a straight line, which intersects the $\alpha - \beta$ plane at the point (α_i, β_i) .

Adopting the celestial coordinates is very convenient to describe the shadow (see for example Vázquez and Esteban (2004)):

$$\alpha = \lim_{r_0 \rightarrow \infty} \left(-r_0^2 \sin \theta_0 \frac{d\phi}{dr} \right), \quad (13)$$

and

$$\beta = \lim_{r_0 \rightarrow \infty} r_0^2 \frac{d\theta}{dr}, \quad (14)$$

since we consider an observer far away from the black hole r_0 goes to infinity, θ_0 is the angular coordinate of the observer, i.e. the inclination angle between the rotation axis of the black hole and the line of sight of the observer. The geometry of the new introduced coordinates is shown in Fig. 2. The coordinates α and β are the apparent perpendicular distances of the image as seen from the axis of symmetry and from its projection on the equatorial plane, respectively.

Calculating $d\phi/dr$ and $d\theta/dr$ from the space-time metric given by expression (1) and taking the limit of a far away observer we obtain celestial coordinates as a function of the constants of motion in the form

$$\alpha = -\xi \csc \theta_0, \quad (15)$$

and

$$\beta = \pm \left[\eta + \left(a^2 - \frac{4l^2}{\sin^2 \theta_0} \right) \cos^2 \theta_0 - \xi^2 \cot^2 \theta_0 + 4l \cos \theta_0 \left(a - \frac{\xi}{\sin^2 \theta_0} \right) \right]^{1/2}, \quad (16)$$

where Eqs. (6), (7), and (8) were used to calculate $d\theta/dr$ and $d\phi/dr$. These equations have implicitly the same form as for the Kerr metric, with the new radial functions ξ and η given by Eqs. (11) and (12) (a detailed calculation of the values of ξ and η , and the expressions of the celestial coordinates α and β as a function of the constants of motion for the Kerr geometry, are given in (Vázquez and Esteban 2004)).

In the case of a Kerr black hole, we may introduce two observables which approximately characterize the apparent shape. First we approximate the apparent shape by a circle passing through three points which are located at the top position (A), the bottom position (B), and the most right end (C) of the shadow as shown by three red points in Fig. 3. The point C corresponds to the unstable retrograde circular orbit when seen from an observer on the equatorial plane. We define the radius R_s of the shadow by the radius of this approximated circle. We also take into account the dent in the left-hand side of the shadow. The size of this dent is evaluated by D_{cs} , which is the difference between the left endpoints of the circle and of the shadow (see Fig. 3). Then we define the distortion parameter δ_s of the shadow by $\delta_s = D_{cs}/R_s$. Thus we adopt these two variables (R_s and δ_s) as observable in astronomical observation (Hioki and Maeda 2009).

When the observer is situated in the equatorial plane of the black hole, the inclination angle is $\theta_0 = \pi/2$ and the gravitational effects on the shadow, which grow with θ_0 , are larger. The inclination angle corresponding to the Galactic supermassive black hole is also expected to lie close to $\pi/2$. In this interesting case, we have simply

$$\alpha = -\xi, \quad (17)$$

and

$$\beta = \pm \sqrt{\eta}. \quad (18)$$

For the visualization of the shape of the black hole shadow one needs to plot β vs α . In Fig. 4, we show the contour of the shadows of black holes. From the plots one can see that the presence of the gravitomagnetic charge will increase the effective size of the shadow. We plot the shapes of the shadow of black hole with the gravitomagnetic charge for the different values of black

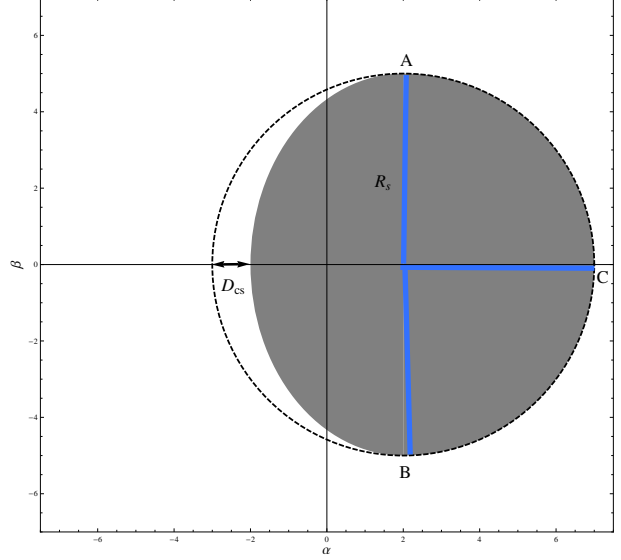


Fig. 3 The observables for the apparent shape of a Kerr black hole are the radius R_s and the distortion parameter $\delta_s = D_{cs}/R_s$, approximating it by a distorted circle, where D_{cs} is the difference between the left endpoints of the circle and of the shadow.

hole angular momentum a : $a/M = 0.5$, $a/M = 0.7$, $a/M = 0.8$, and $a/M = 0.99$. One can compare the effect of the NUT parameter and the black hole angular momentum on modification of the shape of black hole. It appears they have opposite effects on black hole shadow size. The gravitomagnetic charge increases the size of the shadow shape while black hole's angular momentum decreases its size.

The observable R_s can be calculated from the equation

$$R_s = \frac{(\alpha_t - \alpha_r)^2 + \beta_t^2}{2|\alpha_t - \alpha_r|},$$

and the observable δ_s is given by

$$\delta_s = \frac{\tilde{\alpha}_p - \alpha_p}{R_s},$$

where $(\tilde{\alpha}_p, 0)$ and $(\alpha_p, 0)$ are the points where the reference circle and the contour of the shadow cut the horizontal axis at the opposite side of $(\alpha_r, 0)$, respectively. In Fig. 5, the observables R_s and δ_s are shown as functions of the gravitomagnetic charge l . From the dependence of R_s from the NUT parameter one can again see that the gravitomagnetic charge forces to increase the size of the black hole shadow. The dependence of δ_s from NUT charge shows that gravitomagnetic charge forces to shadow to get the shape of circle than ellipse. In the case of rotation, with the increase of black hole's angular momentum the shape of black hole shadow takes form of ellipse rather than circle.

4 PARTICLE CAPTURE CROSS SECTIONS FOR BLACK HOLE WITH NUT CHARGE

In this section we will study the pure effect of NUT parameter assuming that the angular momentum of the black hole is equal to zero. It has been shown in the paper of Abdujabbarov et al. (2008) that variables in the Hamilton-Jacobi equation for the particle motion around NUT black hole can be separated in the equatorial plane. In the space-time metric (1) we assume that the central object is non-rotating and particles are confined at the equatorial plane ($a = 0$, and $\theta = \pi/2$).

It was first shown by Zimmerman and Shahir (1989) for the spherical symmetric case (NUT spacetime) and later in the paper of ? for the axial symmetric case (Kerr-Taub-NUT spacetime) that the orbits of the test particles are confined to a cone with the opening angle θ given by $\cos \theta = 2\mathcal{E}l/\mathcal{L}$. It also follows that in this case the equations of motion on the cone depend on l only via l^2 (?Abdujabbarov et al. 2008).

The main point is that the small value for the upper limit for gravitomagnetic moment has been obtained by comparing theoretical results with experimental data as (i) $l/M \leq 10^{-24}$ from the gravitational microlensing (Rahvar and Habibi 2004), (ii) $l/M \leq 1.5 \cdot 10^{-18}$ from the interferometry experiments on ultra-cold atoms (Morozova and Ahmedov 2009), (iii) and similar limit has been obtained from the experiments on Mach-Zehnder interferometer (Kagramanova et al. 2008) (here M is the total mass of central gravitating object).

Due to the smallness of the gravitomagnetic charge let us consider the motion in the quasi-equatorial plane when the motion in θ direction changes as $\theta = \pi/2 + \delta\theta(t)$, where $\delta\theta(t)$ is the term of first order in l , then it is easy to expand the trigonometric functions as $\sin \theta = 1 - \delta\theta^2(t)/2 + \mathcal{O}(\delta\theta^4(t))$ and $\cos \theta = \delta\theta(t) - \mathcal{O}(\delta\theta^3(t))$. Neglecting the small terms $\mathcal{O}(\delta\theta^2(t))$, one can write $\Sigma = r^2 + l^2$, $\Delta = r^2 - 2Mr - l^2$, and $\chi = 0$ and consequently the equation of motion for the radial motion takes the following form

$$r^4 \left(\frac{dr}{d\lambda} \right)^2 = R(r) = [\mathcal{E}^2 - 1 - 2U_{\text{eff}}(r, l, \mathcal{L})] r^4, \quad (19)$$

where \mathcal{E} and \mathcal{L} is the energy and angular momentum of the particle per unit of its mass and the quantity

$$U_{\text{eff}}(r, l, \mathcal{L}) = -\frac{l^2 + Mr}{\Sigma} + \frac{\Delta \mathcal{L}^2}{2\Sigma^2} \quad (20)$$

can be interpreted as effective potential of the radial motion of the test particle at equatorial plane. The radial dependence of the effective potential of radial

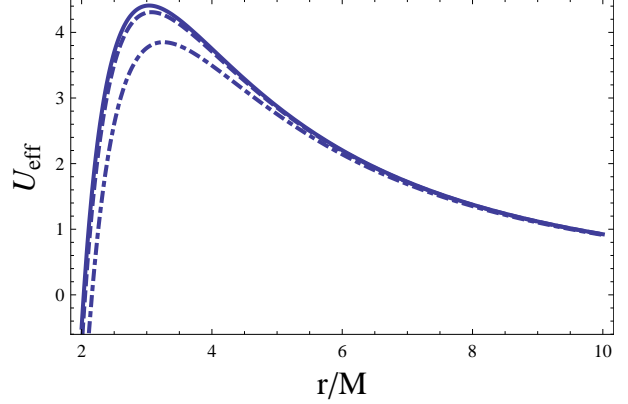


Fig. 6 The radial dependence of the effective potential of radial motion of the massive particles for the different values of the gravitomagnetic charge: solid line for $l/M = 0.1$, dashed line for $l/M = 0.5$, and dot-dashed line for $l/M = 0.9$.

motion of the massive particles for the different values of the gravitomagnetic charge is presented in Fig. 6.

Assuming that the uncharged particle is moving slowly at infinity, i.e. $\mathcal{E} \simeq 1$ one can easily rewrite the expression (19) in the following form:

$$R(\rho) = \rho^3 + (\tilde{l} - \tilde{\mathcal{L}}) \rho^2 + \tilde{\mathcal{L}} \rho + \frac{\tilde{l}\tilde{\mathcal{L}}}{2}, \quad (21)$$

where

$$\rho = \frac{r}{M}, \quad \tilde{l} = \left(\frac{l}{M} \right)^2, \quad \tilde{\mathcal{L}} = \left(\frac{\mathcal{L}}{M} \right)^2.$$

Gravitational capture of the particle occurs for $\mathcal{L} \leq \mathcal{L}_{\text{cr}}$. For the $\mathcal{L} = \mathcal{L}_{\text{cr}}$ orbit spirals into a circular orbit, the radius of which is determined by the value of the multiple root of the polynomial (21), i.e. discriminant of the later should vanish. Now it is easy to find the expression for \mathcal{L}_{cr} as

$$\mathcal{L}_{\text{cr}}^2 = 16M^2 - 6l^2 - \frac{13l^4}{16M^2}. \quad (22)$$

In the Fig.7 the dependence of \mathcal{L}_{cr} from dimensionless NUT parameter is presented. The dependence shows that the presence of the gravitomagnetic charge decreases the capture cross section for particles by black hole.

5 Conclusion

In this paper, we have studied the shadow of black hole with nonvanishing gravitomagnetic charge and analyzed how the shadow of the black hole will be distorted by the presence of the NUT parameter. From

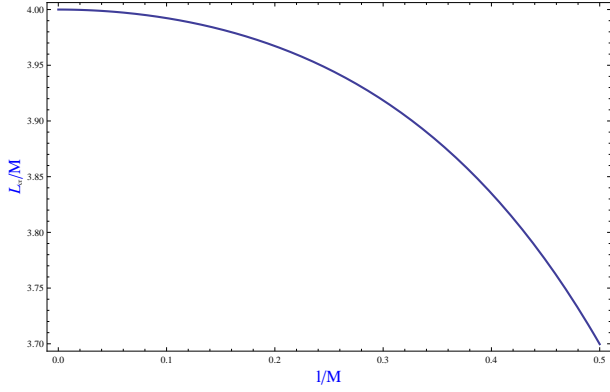


Fig. 7 The dependence of the critical angular momentum for capturing by central black hole from gravitomagnetic monopole momentum.

the numerical results we have obtained that the NUT parameter forces to increase the size of the black hole shadow. The dependence of the distortion parameter δ_s from the NUT charge shows that the gravitomagnetic charge forces black hole's shadow to get the shape of circle than ellipse. We have also studied the capture cross section for massive particles by black hole with non-vanishing gravitomagnetic charge and found its strong dependence from the NUT parameter.

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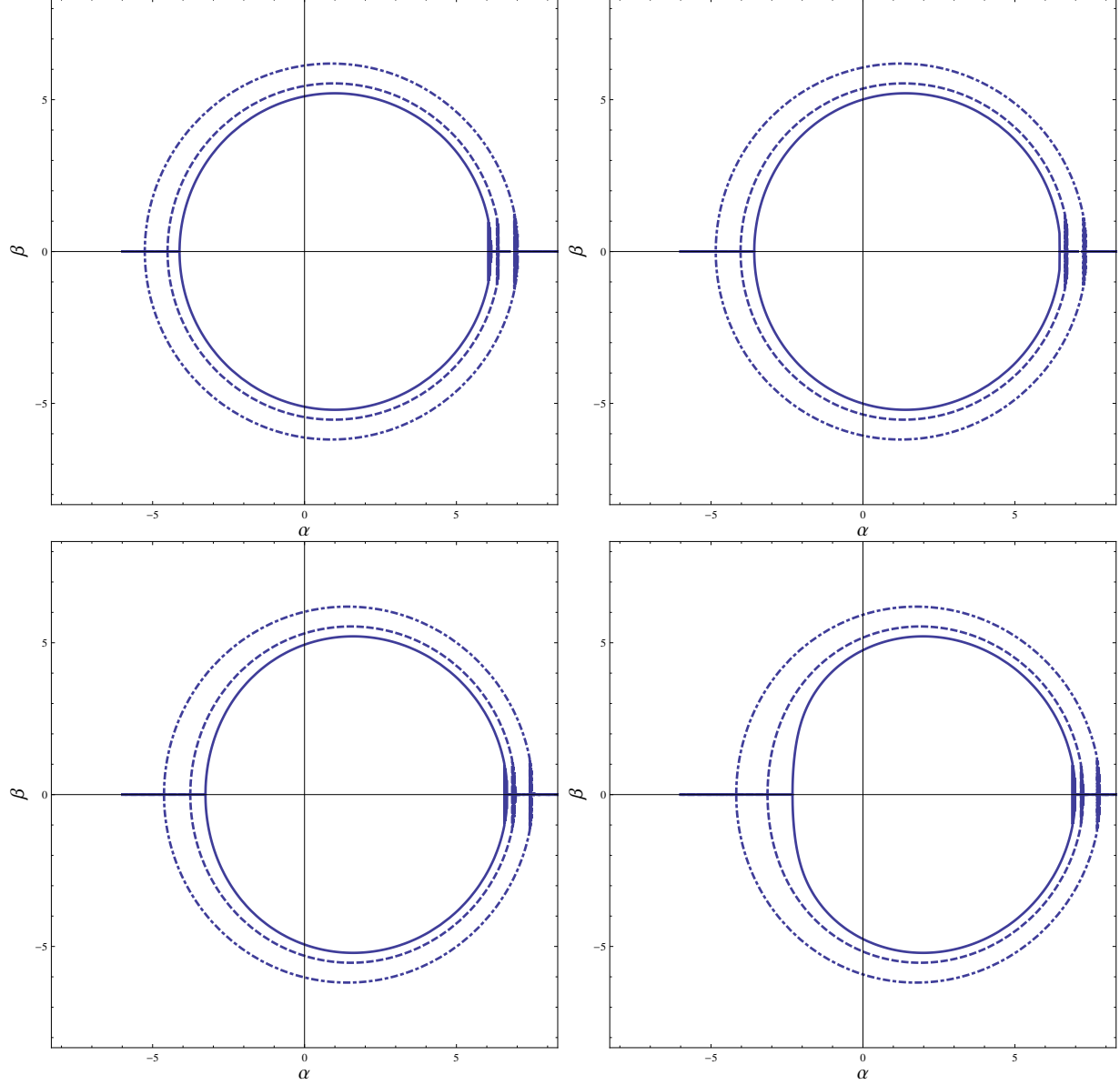


Fig. 4 Silhouette of the shadow cast by a Kerr-Taub-NUT black hole situated at the origin of coordinates with inclination angle $\theta = \pi/2$, having a black hole angular momentum a and a NUT charge l . Upper row, left: $a/M = 0.5$, $l/M = 0.1$ (solid line), $l/M = 0.5$ (dashed line), and $l/M = 0.9$ (dashed-dotted line). Upper row, right: $a/M = 0.7$, $l/M = 0.1$ (solid line), $l/M = 0.5$ (dashed line), and $l/M = 0.9$ (dashed-dotted line). Lower row, left: $a/M = 0.8$, $l/M = 0.1$ (solid line), $l/M = 0.5$ (dashed line), and $l/M = 0.9$ (dashed-dotted line). Lower row, right: $a/M = 0.99$, $l/M = 0.1$ (solid line), $l/M = 0.5$ (dashed line), and $l/M = 0.9$ (dashed-dotted line). The shadow corresponds to each curve and the region inside it.

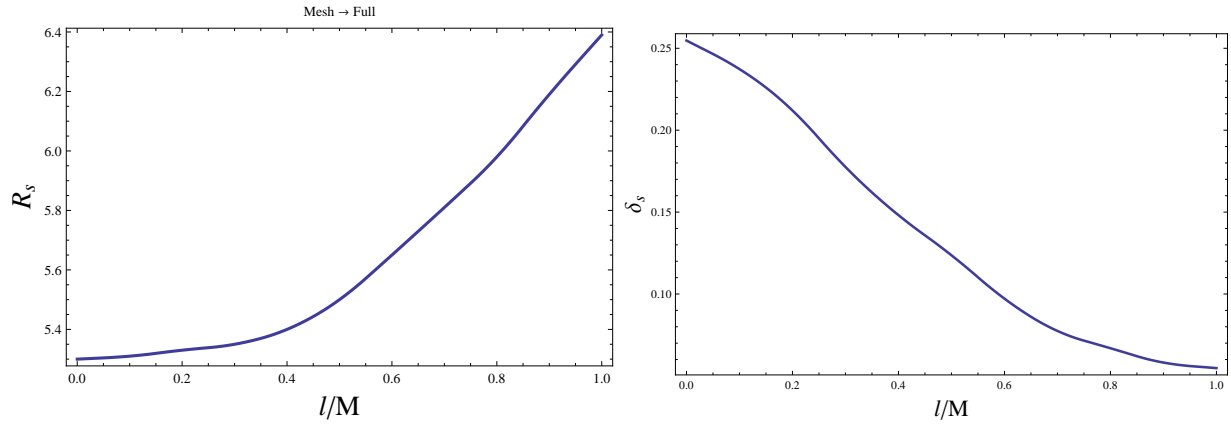


Fig. 5 Observables R_s and δ_s as functions of the NUT charge, corresponding to the shadow of a black hole situated at the origin of coordinates with inclination angle $\theta = \pi/2$ and spin parameters $a = 0.99$.

References

- Abdujabbarov, A.A., Ahmedov, B.J., Kagramanova, V.G.:
Gen. Rel. Grav. **40**, 2515 (2008)
- Abdujabbarov, A.A., Ahmedov, B.J., Shaymatov, S.R.,
Rakhmatov, A.S.: Astrophys Space Sci **334**, 237 (2011)
- Aliev, A. N., Cebeci, H., Dereli, T.: Phys. Rev. D **77**,
124022 (2008)
- Amarilla, L., Eiroa, E. F.: Phys Rev. D **85**, 064019 (2012)
- Connell, P., Frolov, V., Kubizňák, D.: Phys Rev. D **78**,
024042 (2008)
- Dadhich, N., Turakulov, Z.Ya.: Class. Quantum Grav. **19**,
2765 (2002)
- Frolov, V., Krtouš, P.; Phys Rev. D **83**, 024016 (2011).
- Grunau, S., Kagramanova, V.: Phys Rev. D **83**, 044009
(2011)
- Hackmann, E., Lämmerzahl, C.: Phys Rev. D **85**, 044049
(2012)
- Hioki K., Maeda, K.I.: Phys. Rev. D **80**, 024042 (2009)
- Kagramanova, V., Kunz, J., Lämmerzahl, C.: Class. Quan-
tum Grav. **25**, 105023 (2008)
- Kagramanova, V., Kunz, J., Hackmann, E., Lämmerzahl,
C.: Phys Rev. D **81** 124044 (2010)
- Krtouš, P., Frolov, V., Kubizňák, D.: Phys Rev. D **78**,
064022 (2008)
- Kubizňák, D., Frolov, V., Krtouš, P., Connell, P.: Phys Rev.
D **79**, 024018 (2009)
- Morozova, V.S., Ahmedov, B.J.: Int. J. Mod. Phys. D **18**,
107 (2009)
- Morozova, V. S., Ahmedov, B.J., Kagramanova, V.G.: As-
trophys. J. **684**, 1359 (2008)
- Nedkova, P., Yazadjiev, S.: Phys Rev. D **85**, 064021 (2012)
- Newman, E., Tamburino, L., Unti, T.: J. Math. Phys. **4**,
915 (1963)
- Nouri-Zonoz, M.: Class. Quantum Grav. **21**, 471 (2004)
- Rahvar, S., Habibi, F.: Astroph. J., **610**, 673 (2004)
- Virmani, A.: Phys Rev. D **84**, 064034 (2011)
- Vázquez S., Esteban, E.: Nuovo Cim. **119B**, 489 (2004)
- Zimmerman, R., Shahir, B.: Gen. Relativ. Gravit. **21**, 821
(1989)